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The Reunions of Three Dissimilar Vicious Walkers

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We study the behavior of three "vicious" random walkers which diffuse freely in one dimension with *arbitrary* diffusivities b_1^2, b_2^2, b_3^2, except that their paths may not cross. The full distribution function is calculated exactly in the continuum limit; the exponent ψ_3 governing the decay of the probability $R_n^{(3)} \sim 1/n^{\psi_3}$ of a simultaneous *reunion* of all three walkers after n steps is found to vary *continuously* according to $\psi_3 = 1 + \pi/\cos^{-1}\{b_2^2/[(b_1^2+b_2^2)(b_2^2+b_3^2)]^{1/2}\}$. This variation has consequences for various interfacial wetting transitions in $(1+1)$ dimensions. It may also be related heuristically to the marginality of direct interface–wall interactions decaying as W_0/l^2 in the intermediate fluctuation regime of $(1+1)$-dimensional wetting, where exponents varying continuously with W_0 have recently been found.

KEY WORDS: Random walkers; wetting transitions; vicious drunks; continuously variable exponents; reunions.

1. INTRODUCTION

Consider p random walkers who walk on a line, the coordinate of the jth walker being x_j $(j = 1, 2,..., p)$. Following ref. 1 (to be referred to as I), we may suppose, for concreteness, that at each tick of a clock an isolated walker takes a step of length a to the left or the right with statistical weight $w_j^+ = w_j^-$ or rests on the same site with weight w_j^0. The mean square step length

$$b_j^2 = 2w_j^+ a^2/(w_j^0 + 2w_j^+) \tag{1.1}$$

specifies the diffusivity b_j^2 of the jth walker. When all p walkers are present we suppose[1] that at each tick of the clock a walker is randomly chosen to

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move (or to rest); all walkers move on the same lattice of sites l, located at $x = la$. (One may consider also a "lock-step" dynamics: see I.)

Vicious walkers[1] are short-sighted, experiencing no interactions when on different sites; however, when two arrive on the *same* site, they shoot each other dead, leaving two fewer walkers! (After such a death the whole process may be regarded as terminated.) The obvious questions are: What is the probability that all p walkers survive for n steps starting from some initial configuration $x_j = x_{j,0}$ ($j = 1,..., p$)?; What is the properly weighted spatial distribution function or partition function

$$Q_n^{(p)}(\mathbf{x}, \mathbf{x}_0), \qquad \mathbf{x} = (x_1,..., x_p), \qquad \mathbf{x}_0 = (x_{1,0},..., x_{p,0}) \tag{1.2}$$

of the p survivors after n steps? Finally, it was shown in I that, for applications to a variety of physical problems in *two* spatial dimensions, the most interesting question relates to the probability of a *reunion* in which all walkers start close to (but not at) the origin, say, at spacing a apart, and after n steps *all* meet close together again at a mean position $\bar{x}$. The probability of a reunion anywhere was found to decay as

$$R_n^{(p)} \approx C_p/n^{\psi} \qquad \text{when} \qquad n \to \infty \tag{1.3}$$

and it was shown[1] that the exponent ψ determined the order of a variety of interfacial wetting transitions, the existence and nature of the singular corrections to various commensurate–incommensurate transitions, and the decay of correlations in ordered states. Typically, the trajectories, $x_j(n)$ of the individual walkers in the (x, n) plane represent fluctuating interfaces between distinct phases or domains in two spatial dimensions (x, y). The diffusivity b_j^2 corresponds to the interfacial stiffness $\tilde{\Sigma}(T)$ of the jth interface or domain wall.[1,2]

In I these problems were solved and analyzed in detail for the case of p *similar* vicious walkers with $b_j = b$ (all j). Specifically, it was shown by the method of images applied to an associated diffusion problem that the overall distribution was given explicitly by

$$Q_n^{(p)}(\mathbf{x}, \mathbf{x}_0) = \sum_{\hat{\pi}} (-)^{|\hat{\pi}|}\, Q_n^0(\mathbf{x}, \hat{\pi}\mathbf{x}_0) \tag{1.4}$$

where $\hat{\pi}\mathbf{x}_0$ denotes a permutation of parity $|\hat{\pi}| = 0, 1$ of the initial coordinates $(x_{1,0}, x_{2,0},..., x_{p,0})$, while the noninteracting or "harmless" overall distribution is

$$Q_n^0(\mathbf{x}, \mathbf{x}_0) = \prod_{j=1}^{p} Q_n^0(x_j, x_{j,0}) \tag{1.5}$$

in which

$$Q_n^0(x, x_0) \approx e^{-\sigma n} e^{-(x-x_0)^2/2b^2 n}/(2\pi b^2 n)^{1/2} \tag{1.6}$$

represents the distribution for a single, isolated walker of diffusivity b^2 and total step weight $e^{-\sigma} \equiv w^0 + 2w^+$ (where σ corresponds to the reduced interfacial tension $\Sigma/k_B T$). From (1.4) one can obtain[1,3,4] the result

$$\psi_p = \tfrac{1}{2}(p^2 - 1) = 0, \tfrac{3}{2}, 4, 7\tfrac{1}{2},... \quad \text{for} \quad p = 1, 2, 3, 4,... \tag{1.7}$$

for the reunion exponent for p similar walkers. A variety of other explicit formulas follow.[1] Attention should also be drawn to recent work by Duplantier,[5] who shows how to treat a range of intersection problems for similar random walkers in general dimensionalities by renormalization group methods.[2]

Two other vicious walk problems were addressed in I. The first concerned p walkers in the presence of a rigid *absorbing wall* (or "death-dealing cliff") fixed at the origin $x = 0$, the walkers being confined to, say, $x > 0$. The analogous reunion problem for walkers starting and finishing close to the wall (i.e., $\bar{x} = 0$) was solved for $p = 1$ and 2 *similar* walkers ($b_j = b$), leading to the exponents

$$\psi_1^W = \tfrac{3}{2}, \qquad \psi_2^W = 5 \tag{1.8}$$

and corresponding expressions for the dependence of the decay amplitudes C_p^W on the actual initial and final conditions.

The second problem concerned *dissimilar walkers* with distinct diffusivities b_j^2. This is clearly of interest since, in the absence of some special physical symmetry, successive interfaces separating different spatially extended phases A, B, C,..., will normally have different tensions and stiffnesses. Indeed, one might reasonably regard a rigid, fixed absorbing wall as simply an infinitely stiff interface to be described by a vicious walker of vanishing diffusivity, $b_i^2 \to 0$.

Now the method of images used in I fails for *dis*similar walkers. However, progress can be made for $p = 2$ walkers if one first goes to the continuum or diffusive limit, which, in fact, suffices for the asymptotic

[2] See also the references in Duplantier's article[5] which list recent work by G. F. Lawler (Duke University) treating the probability of intersection of three walkers in $d = 3$ dimensions, which is a borderline for this problem. D. ben-Avraham, *J. Chem. Phys.* **88**:941 (1988), quotes, in his Eq. (7), a result for the survival probability for three vicious walkers equivalent to our (3.6) in the restricted case $b_1 = b_3$; he thanks François Leyvraz for assistance. We are grateful to Professor S. Redner for telling us of this work.

properties of interest to us. In this limit the distribution function for the set of p walkers satisfies the equation

$$\frac{\partial Q^{(p)}}{\partial t}(\mathbf{x}, t) = \tfrac{1}{2} \sum_{j=1}^{p} b_j^2 \frac{\partial^2 Q^{(p)}}{\partial x_j^2} - \sigma_{\text{tot}} Q^{(p)} \tag{1.9}$$

where, for the sake of convention, we have replaced n by t [and $Q_n^{(p)}(\mathbf{x})$ by $Q^{(p)}(\mathbf{x}, t)$], while

$$\sigma_{\text{tot}} = \sum_{j=1}^{p} \sigma_j \qquad \text{with} \quad e^{-\sigma_j} = w_j^0 + 2w_j^+ \tag{1.10}$$

The solution for a single, isolated walker the standard form on the right side of (1.6). The character of *viciousness* is then embodied in the constraint

$$x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots \leqslant x_p \tag{1.11}$$

with boundary conditions, for all t,

$$Q(\mathbf{x}, t) \to 0 \qquad \text{as} \quad x_j \to x_{j+1} \quad (j = 1, 2, \ldots, p-1) \tag{1.12}$$

For $p = 2$ walkers the change of coordinates from (x_1, x_2) to $(x, \bar{x})$ with

$$x \equiv x_{12} = x_2 - x_1 \geqslant 0, \qquad \bar{x} = \tfrac{1}{2}[(b_2/b_1)\, x_1 + (b_1/b_2)\, x_2] \tag{1.13}$$

separates the diffusion equation (1.9); the "external walker," with coordinate $\bar{x}$, diffuses freely; the "internal walker," with coordinate x, can be dealt with by images.[1] The result may be written

$$Q^{(2)}(\mathbf{x}, \mathbf{x}_0, t) = \frac{\exp[-(\sigma_1 + \sigma_2)\, t] \exp[-(\bar{x} - \bar{x}_0)^2/\bar{b}^2 t]}{2\pi \bar{b}^2 t} \times \left[1 - \exp\left(-\frac{x x_0}{\bar{b}^2 t}\right)\right] \exp\left[-\frac{(x - x_0)^2}{4\bar{b}^2 t}\right] \tag{1.14}$$

where the mean diffusivity is just

$$\bar{b}^2 = \tfrac{1}{2}(b_1^2 + b_2^2) \tag{1.15}$$

To find the probability of a reunion anywhere, one should integrate on $\bar{x}$, which removes a factor $t^{-1/2}$. Then, replacing t by n and letting $n \to \infty$ at fixed x and x_0 yields the form (1.3) with $\psi_2 = 3/2$ and

$$C_2 = (x_2 - x_1)(x_{2,0} - x_{1,0})/2\pi^{1/2} \bar{b}^3 \tag{1.16}$$

Thus, the exponent ψ is the *same* as for two similar walkers: see (1.7). Likewise, the dependence of the amplitude C_2 on $\mathbf{x}$ and $\mathbf{x}_0$ is found to be identical up to a factor.[4] Furthermore, if one lets $b_1^2 \to 0$, so that the first walker becomes a rigid wall at $x_1 = 0$, one recaptures from (1.14) the exact (continuum) result for a wall plus one walker with $\psi_1^W = 3/2$, as in (1.8), and the correct value of the amplitude $C_1^W \equiv C_2$.

The case of $p = 3$ *dissimilar* walkers, however, presents a harder problem which was not addressed in I. It was observed, however, that the behavior as a function of b_1, b_2, and b_3 had to be more complex, since the exponent ψ for three similar walkers takes the value

$$\psi_3(b_1 = b_2 = b_3 > 0) = 4 \tag{1.17}$$

as shown in (1.7), whereas if $b_1 \to 0$ so as to yield a rigid wall and two similar walkers, one has

$$\psi_3(b_1 = 0; b_2 = b_3 > 0) = \psi_2^W = 5 \tag{1.18}$$

[see (1.8)]. To this we can add

$$\psi_3(b_1, b_3 > 0; b_2 = 0) = 2\psi_1^W = 3 \tag{1.19}$$

since, if the middle walker ceases to diffuse, as $b_2 \to 0$, the two outer walkers clearly become decoupled and one is left with two separate problems of a walker plus a fixed wall.

These contrasting results raise several questions: Must ψ_3 always be an integer? Can it have a value less than 2? Such a low value would imply a continuous wetting transition for the unbinding of one interface $A|||D$ between phases[1,2] A and D into three distinct interfaces $A|B$, $B|C$, and $C|D$, where B and C are intermediate coexisting phases. Can ψ_3 depend continuously on the b_j? And, if so, precisely how? In this paper we analyze and fully solve (for the continuum limit) the problem of three dissimilar vicious walkers. In particular, we obtain an explicit formula for $\psi_3(b_1, b_2, b_3)$ [see (3.1) below] which shows that ψ_3 varies *continuously*, a surprising result! The limiting cases (1.17)–(1.19) are trivially reproduced by the formula. The general expression for the amplitude $C_3(\mathbf{x}, \mathbf{x}_0; b_1, b_2, b_3)$ likewise reproduces the differing forms found previously[1] for the cases (1.17)–(1.19). These results follow from the expression (2.28) below for the full distribution $Q^{(3)}(\mathbf{x}, \mathbf{x}_0; t)$.

2. THE DISTRIBUTION FOR THREE DISSIMILAR WALKERS

As a first step in the analysis of three vicious walkers we may define $G(\mathbf{x}, \mathbf{x}_0; t)$ via

$$Q^{(3)}(\mathbf{x}, \mathbf{x}_0; t) = e^{\sigma_{\text{tot}} t} G(\mathbf{x}, \mathbf{x}_0; t) \tag{2.1}$$

so that G satisfies the anisotropic equation (1.9) with $\sigma_{\text{tot}} \equiv 0$. If we set

$$x_j = b_j y_j, \qquad x_{j,0} = b_j y_{j,0} \qquad (j = 1, 2, 3) \tag{2.2}$$

then $G \equiv G(\mathbf{y}, \mathbf{y}_0; t)$ solves the spatially isotropic diffusion equation

$$\frac{\partial G}{\partial t} = \frac{1}{2} \sum_{j=1}^{3} \frac{\partial^2 G}{\partial y_j^2} \equiv \frac{1}{2} \nabla_y^2 G \tag{2.3}$$

The condition that the walkers not meet, embodied in (1.11) and (1.12), requires

$$b_1 y_1 \leqslant b_2 y_2 \leqslant b_2 y_3 \tag{2.4}$$

with $G(y_1, y_2, y_3; t) \to 0$ for all t as $\mathbf{y}$ approaches the planes

$$(+) \qquad \hat{\mathbf{n}}_+ \cdot \mathbf{y} \propto -b_2 y_2 + b_3 y_3 = 0 \tag{2.5}$$

$$(-) \qquad \hat{\mathbf{n}}_- \cdot \mathbf{y} \propto b_1 y_1 - b_2 y_2 = 0 \tag{2.6}$$

Here $\hat{\mathbf{n}}_+$ and $\hat{\mathbf{n}}_-$ are unit vectors normal to the planes $(+)$ and $(-)$; explicitly, we have

$$\hat{\mathbf{n}}_- = (b_1, -b_2, 0)/(b_1^2 + b_2^2)^{1/2}, \qquad \hat{\mathbf{n}}_+ = (0, -b_2, b_3)/(b_2^2 + b_3^2)^{1/2} \tag{2.7}$$

These two bounding planes intersect and include an angle $\Theta(b_1, b_2, b_3)$ given by

$$\cos \Theta = \hat{\mathbf{n}}_+ \cdot \hat{\mathbf{n}}_- = b_2^2/[(b_1^2 + b_2^2)(b_2^2 + b_3^2)]^{1/2} \tag{2.8}$$

Note that $\Theta \to 0$ when $b_1, b_3 \to 0$, as correct, since the $(+)$ and $(-)$ planes then degenerate into the plane $y_2 = 0$. One can also write

$$\Theta(b_1, b_2, b_3) = \tan^{-1}(\beta_1^2 + \beta_3^2 + \beta_1^2 \beta_3^2)^{1/2} \qquad \text{with} \quad \beta_j = b_j/b_2 \tag{2.9}$$

Evidently one has $0 \leqslant \Theta \leqslant \pi/2$.

The intersection axis of the planes $(+)$ and $(-)$ is characterized by a unit vector

$$\hat{\mathbf{Z}} = \hat{\mathbf{n}}_+ \wedge \hat{\mathbf{n}}_- / \sin \Theta = \tilde{b}(b_1^{-1}, b_2^{-1}, b_3^{-1}) \tag{2.10}$$

where

$$\tilde{b} = \frac{b_1 b_2 b_3}{(b_1^2 b_2^2 + b_2^2 b_3^2 + b_3^2 b_1^2)^{1/2}} = \left(\sum_{i=1}^{3} b_i^{-2} \right)^{-1/2} \tag{2.11}$$

Next note that from the unit vector

$$\hat{\mathbf{Y}} \equiv \hat{\mathbf{n}}_- \tag{2.12}$$

which is perpendicular to $\hat{\mathbf{Z}}$, we can construct a unit vector

$$\hat{\mathbf{X}} = \hat{\mathbf{Z}} \wedge \hat{\mathbf{Y}} = \frac{\tilde{b}}{(b_1^2 + b_2^2)^{1/2}} \left(\frac{b_2}{b_3}, \frac{b_1}{b_3}, -\frac{b_1}{b_2} - \frac{b_2}{b_1} \right) \tag{2.13}$$

orthogonal to $\hat{\mathbf{Z}}$ and $\hat{\mathbf{Y}}$ and so lying in the $(-)$ plane. Using this new orthonormal basis, we can introduce cylindrical coordinates (z, r, θ) via

$$z(\mathbf{x}) = \sum_{j=1}^{3} \tilde{b} x_j / b_j^2 \tag{2.14}$$

$$r^2(\mathbf{x}) = \sum_{j=1}^{3} \frac{x_j^2}{b_j^2} - z^2 = \left(1 - \frac{\tilde{b}^2}{b_1^2}\right) \frac{x_1^2}{b_1^2} - 2\tilde{b}^2 \frac{x_1 x_2}{b_1^2 b_2^2} + \cdots \tag{2.15}$$

$$\tan \theta = \frac{b_1 b_2 b_3 (x_2 - x_1)}{\tilde{b}[b_1^2 (x_3 - x_2) + b_2^2 (x_3 - x_1)]} \tag{2.16}$$

Finally, we can rewrite the basic diffusion equation (2.3) and the conditions (2.4)–(2.6) in cylindrical coordinates to obtain

$$\frac{\partial G(z, r, \theta)}{\partial t} = \frac{1}{2} \left(\frac{\partial^2 G}{\partial z^2} + \frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} + \frac{1}{r^2} \frac{\partial^2 G}{\partial \theta^2} \right) \tag{2.17}$$

with $-\infty < z < \infty$, $0 \leqslant r < \infty$, and

$$0 \leqslant \theta \leqslant \Theta \tag{2.18}$$

together with

$$G \to 0 \quad \text{as} \quad z \to \pm\infty, \quad r \to 0, \infty, \quad \text{or} \quad \theta \to 0, \Theta \tag{2.19}$$

In other words, we have diffusion in an absorbing wedge with axis along the z axis and opening angle Θ.[3] We want the corresponding Green's function for a unit point source located at (z_0, r_0, θ_0), which can be computed from the initial coordinates $(x_{1,0}, x_{2,0}, x_{3,0})$ of the walkers on the line via (2.14)–(2.16). This is a standard but nontrivial problem in the theory of heat conduction. We will quote below the full solution in the form presented by Carslaw and Jaeger.[(6)] However, it turns out that the asymptotic features of most interest to us can be extracted by elementary means. Accordingly, we proceed in a straightforward fashion.

[3] Incidentally, the fact that ψ should not depend on the b_j for only two dissimilar walkers is evident, following the transformation (2.2), in that the change of variables then leaves the boundary surface as a simple line with no intrinsic b_j-dependent geometrical features, in contrast to the angle Θ in (2.8) for $p = 3$ walkers and further such angles for $p > 3$.

It is evident from (2.17) that the z motion separates off. Indeed, this coordinate, which, in essence, is just the center of mass of the three walkers on the line, diffuses freely. Thus, we have

$$G(\mathbf{y}, \mathbf{y}_0; t) = P(r, \theta; r_0, \theta_0; t)\{\exp[-(z-z_0)^2/2t]\}/(2\pi t)^{1/2} \quad (2.20)$$

where P is the Green's function of the equation for diffusion in the planar wedge $(0 \leqslant r < \infty, 0 \leqslant \theta \leqslant \Theta)$. The angular part of this latter equation clearly has solutions of the form $\sin l\nu\theta$ which satisfy the boundary conditions (2.19) provided

$$\nu = \pi/\Theta, \qquad l = 1, 2,... \quad (2.21)$$

Note that the inequality $\nu \geqslant 2$ follows from (2.9). On invoking the symmetry of the Green's function under interchange of $\mathbf{y}$ and $\mathbf{y}_0$ or of (r, θ) and (r_0, θ_0), we can thus write

$$P(t) = \sum_{l=1}^{\infty} R_l(r, r_0; t) \sin l\nu\theta \sin l\nu\theta_0 \quad (2.22)$$

where $R_l(r, r_0; t)$ is a solution of

$$\frac{\partial R}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \frac{l^2\nu^2}{r^2} R\right) \quad (2.23)$$

Now the homogeneity of this equation in r and t shows that the required Green's function is of the scaling form

$$R_l(r, r_0; t) = t^{-1} U_l(r/t^{1/2}, r_0/t^{1/2}) \quad (2.24)$$

with $U_l(v, v_0) = U_l(v_0, v)$. We may seek power-law solutions,

$$U_l(v, v_0) = v^{\zeta} \sum_{m=0} U_{l,m}(v_0)\, v^m \quad (2.25)$$

which should be valid for small r (or large t as we desire). On substituting in (2.23), the indicial equation is found to be

$$\zeta(\zeta - 1) + \zeta - l^2\nu^2 = 0 \qquad \text{or} \quad \zeta = l\nu \quad (2.26)$$

the solution $-l\nu$ being rejected, since it fails to satisfy (2.19) for $r \to 0$. By (2.9) and (2.24) we now see that ν and hence the decay exponent ψ will vary *continuously* with the b_j. By the symmetry we must have $U_{l,0}(v_0) \approx u_{l,0} v_0^{l\nu}$ as $v_0 \to 0$, where $u_{l,0}$ is a constant. Finally, taking $l = 1$, we conclude that the asymptotic form of the full Green's function as $t \to \infty$ with fixed r and r_0, or, more generally, with r^2/t, $r_0^2/t \to 0$, is given by

$$G(\mathbf{x}, \mathbf{x}_0; t) \approx u_{1,0} \frac{\exp[-(z-z_0)^2/2t]}{(2\pi)^{1/2}\, t^{3/2}} \left(\frac{rr_0}{t}\right)^{\pi/\Theta} \times \sin\frac{\pi\theta}{\Theta} \sin\frac{\pi\theta_0}{\Theta} \tag{2.27}$$

The correction factor to this result is easily seen to be $[1 + O(r^2/t, r_0^2/t)]$.

Our asymptotic analysis is checked by the exact solution of (2.17)–(2.19) given by Carslaw and Jaeger.[6] For the full distribution this yields our principal result

$$Q^{(3)}(\mathbf{x}, \mathbf{x}_0; t) = \frac{4v(b_1, b_2, b_3)}{(2\pi t)^{3/2}} \times \exp[-(\sigma_1 + \sigma_2 + \sigma_3)\, t] \exp\left[-\frac{(z-z_0)^2}{2t} - \frac{r^2 + r_0^2}{2t}\right] \times \sum_{l=1}^{\infty} I_{lv}\left(\frac{rr_0}{t}\right) \sin(lv\theta) \sin(lv\theta_0) \tag{2.28}$$

where, to recapitulate, $v \equiv \pi/\Theta(b_1, b_2, b_3)$ is given by (2.9), the coordinates $(z, r, \theta)(\mathbf{x})$ and $(z_0, r_0, \theta_0)(\mathbf{x}_0)$ follow from (2.14)–(2.16) with (2.11), while the modified Bessel function may be defined by

$$I_v(w) = (\tfrac{1}{2}w)^v \sum_{k=0}^{\infty} (\tfrac{1}{4}w^2)^k / k!\, \Gamma(v+k+1) \tag{2.29}$$

Comparison with (2.27) gives

$$u_{1,0} = 2^{1-v} v / \pi \Gamma(v+1) \tag{2.30}$$

It is worth noting that the second exponential factor in (2.28) can be written symmetrically as

$$\exp\left[-\frac{\bar{b}^2}{2t} \sum_{j=1}^{3} \left(\frac{x_j^2}{b_j^4} + \frac{x_{j,0}^2}{b_j^4}\right) + \frac{\bar{b}^2}{t} \sum_{j,k=1}^{3} \frac{x_j x_{k,0}}{b_j^2 b_k^2}\right] \tag{2.31}$$

with, by (2.11), $\bar{b}^2 = (\sum_j b_j^{-2})^{-1}$. It is also important to recall from (2.16) that θ and θ_0 depend only on the interwalker spacings

$$x_{jk} = x_k - x_j, \qquad x_{jk,0} = x_{k,0} - x_{j,0} \tag{2.32}$$

respectively. The same is true for r and r_0, since one finds

$$r^2 = \frac{x_{12}^2}{b_1^2} + \frac{x_{23}^2}{b_3^2} - \tilde{b}^2 \left(\frac{x_{12}}{b_1^2} - \frac{x_{23}}{b_3^2} \right)^2 \tag{2.33}$$

and similarly for r_0. Furthermore, when the initial and final spacings are the same, that is,

$$x_1 - x_{1,0} = x_2 - x_{2,0} = x_3 - x_{3,0} = \bar{x} \tag{2.34}$$

as in a symmetric reunion, one simply has

$$z - z_0 = \bar{x}/\tilde{b} \tag{2.35}$$

This completes the derivation of the distribution function for three dissimilar vicious random walkers. In the next section we discuss the result and examine various special cases.

3. REUNIONS OF DISSIMILAR WALKERS

The general distribution function (2.28) enables one to answer any question about three dissimilar vicious walkers (in the continuum limit). Consider, then, the probability of a *reunion* anywhere. To this end, (2.28) should be integrated over z at fixed $x_{12} \equiv x_2 - x_1$ and $x_{23} \equiv x_3 - x_2$; this removes a factor $t^{1/2}$ from the denominator and, with $n \equiv t$, yields the power law (1.3) with exponent

$$\psi_3 = 1 + \frac{\pi}{\Theta} = 1 + \frac{\pi}{\tan^{-1}[(b_1^2 b_2^2 + b_2^2 b_3^2 + b_3^2 b_1^2)^{1/2}/b_2^2]} \tag{3.1}$$

The amplitude, after normalization, is found to be

$$C_3(\mathbf{x}, \mathbf{x}_0; b_j) = \frac{2(rr_0/2)^{\pi/\Theta}}{\Theta \Gamma[1 + (\pi/\Theta)]} \sin \frac{\pi\theta}{\Theta} \sin \frac{\pi\theta_0}{\Theta} \tag{3.2}$$

where, by (2.33) and (2.16), r and θ depend only on the spacings x_{12} and x_{23} and likewise for r_0 and θ_0.

The continuous variation of $\psi_3(b_1, b_2, b_3)$ is obvious from (3.1): *non-integral values are generic*. It is easy to check the various previously known special cases mentioned in the Introduction. In the symmetric situation $b_1 = b_2 = b_3$ one finds $\Theta = \pi/3$ and thus $\psi_3 = 4$ as in I. The amplitude reduces to

$$C_3(b_1 = b_2 = b_3) = x_{12} x_{23} x_{31} x_{12,0} x_{23,0} x_{31,0}/4\pi b^6 \tag{3.3}$$

which also checks I. Taking the limit $b_1 \to 0$ and $x_1 \to 0$ yields a rigid wall at the origin plus two walkers at x_2 and x_3. For $b_2 = b_3$ one obtains $\Theta = \pi/4$ and hence $\psi_3 = 5$ which, as mentioned, agrees with the exponent ψ_2^W found in I; similarly, the amplitude

$$C_3(b_1 \to 0, b_2 = b_3) = x_{12} x_{13} x_{12,0} x_{13,0} (x_{13}^2 - x_{13}^2)(x_{13,0}^2 - x_{13,0}^2)/3\pi b^8 \tag{3.4}$$

agrees identically with the result in I. (In deriving this, one notices $\tilde{b}/b_j \to 1$ as $b_j \to 0$ with $b_k > 0$ for $k \neq j$.)

Again, one can take $b_3 \to 0$ and $x_2 \to 0$ to obtain a thin, rigid wall at $x_2 = 0$ with two dissimilar walkers, one at $x_1 < 0$ and one at $x_3 > 0$, which must be quite independent of one another. This limit yields $\Theta = \pi/2$ and so, as anticipated in the Introduction, one has $\psi_3 = 3 = 2\psi_1^W$. The corresponding amplitude is

$$C_3(b_2 = 0) = 2 x_1 x_{1,0} x_3 x_{3,0} / \pi b_1^2 b_3^2 \tag{3.5}$$

which obligingly factorizes into two parts, each confirming the amplitude obtained in I for one walker near a wall!

The value $\psi_3(b_2 = 0) = 3$ is clearly the smallest value that can follow from (3.1). This answers the question posed in the Introduction: consequently, the direct unbinding of an interface $A \| D$ via fluctuations involving compound bubbles bounded by $A | B$ and $C | D$ interfaces and containing a $B | C$ interface should always proceed via a first-order transition (see I). Nevertheless, a finite exponent ψ_3 leads to nontrivial power-law corrections varying as $(T - T_0)^{\psi_3 - 1}$ in the interfacial energy on the bound side of the transition.[1] It should be mentioned, however, that a general treatment of the unbinding of an $A \| D$ interface should also allow for partial (and full) unbinding into the states $A \| C | D$ and $A | B \| D$, as well as into $A | B | C | D$, as described by the simple $p = 3$ reunions. The full distribution $Q^{(3)}(b_1, b_2, b_3)$ given in (2.28), together with the distributions $Q^{(2)}(b_{12}, b_3)$ and $Q^{(2)}(b_1, b_{23})$ following from (1.14) and (1.15), provide a basis for such a treatment. (In an obvious notation, b_{12}^2 and b_{23}^2 are the diffusivities of the interfaces $A \| C$ and $B \| D$, respectively.) The problem is very complicated, since all sequences of intermediate states must be summed to construct the "basic bubble" on the $A \| D$ interface: it will not be attempted here.

If one integrates $Q^{(3)}(\mathbf{x}, \mathbf{x}_0)$ over x_{23} and x_2 with x_{12} (and $\mathbf{x}_0$) fixed, one can calculate the probability of a *partial reunion* in which the first two walkers meet closely on the nth step while the third walker may be anywhere. These two integrations are equivalent to integrating (2.28) on z and on $r \sim x_{23}$; this cancels factors $t^{1/2}$ and $t^{1/2 + \nu/2}$ for $r_0/t^{1/2}$ small. Hence, the partial reunion probability decays with an exponent $\psi_{2,1} = \frac{1}{2}[1 + (\pi/\Theta)]$. A further integration over x_{12} removes another factor

$t^{1/2}$ and leads to the decay law for the total weighted distribution; equivalently, the probability that all three vicious walkers survive for n steps varies as

$$P_n^{(3)}(b_1, b_2, b_3) \sim 1/n^{\pi/2\Theta(b_1,b_2,b_3)} \tag{3.6}$$

as $n \to \infty$. For three similar walkers ($b_1 = b_2 = b_3$) this yields $1/n^{3/2}$, which agrees with the general result $P_n^{(p)}(b_j = b) \sim 1/n^{p(p-1)/4}$ obtained in I.

4. THE INTERMEDIATE FLUCTUATION REGIME IN TWO-DIMENSIONAL WETTING

Finally, it is worth commenting on the continuous variation of $\psi_3(b_j)$ in the light of a recent treatment of the critical wetting or unbinding transition in $d = 2$ dimensions by Lipowsky and Nieuwenhuizen (LN)[7] in the so-called *intermediate fluctuation regime*. Specifically, they address the case of an interface between two coexisting phases which interacts with a rigid wall via a long-range potential $V(l)$ decaying as $1/l^2$; this law is, in fact, *marginal* for the problem, i.e., *on* a borderline of critical behavior between the so-called weak and strong fluctuation regimes.[2,8] The model interface also experiences a short-range attractive interaction, the total LN potential being[7]

$$\begin{aligned} V(l) &= +\infty, && l < 0 \\ &= -U_0, && 0 < l < l_0 \\ &= V_0/l^2, && l > l_0 \end{aligned} \tag{4.1}$$

In the cases of interest here, U_0, which may be regarded as controlled by the temperature T, is positive and, if sufficiently large, serves to bind the interface to the wall even when the long-range interaction is repulsive, i.e., $V_0 > 0$. LN discovered three subregimes of behavior. In subregime (C), which extends to large V_0, the unbinding transition displays the following features:

(a) There is a finite latent heat, so that, by most definitions, the transition is *first order*[1] (although LN do not actually make that characterization).

(b) The gap in the spectrum of the transfer operator for fluctuations along the interface vanishes linearly as $t \equiv U_0(T) - U_c \to 0+$. This would normally be taken as indicating a *critical* transition with a *longitudinal* correlation length $\xi^{\parallel}(T)$ diverging as $1/t^{\nu^{\parallel}}$ with $\nu^{\parallel} = 1$.

(c) The *transverse* correlation lengths defined via the moments

$$\xi_m^{\perp} = \langle [l - \langle l \rangle]^m \rangle^{1/m} \tag{4.2}$$

remain finite as $t \to 0+$ for $m < m_c(V_0)$, where

$$(m_c + 2)^2 = 1 + 8\tilde{\Sigma} V_0 / k_B^2 T^2 > 4 \tag{4.3}$$

However, when $m > m_c$ they diverge with exponents

$$v_m = (m - m_c)/2m \tag{4.4}$$

which, evidently, vary *continuously* with V_0.

To relate these observations to the problem of three vicious random walkers, we return to the $(p = 3)$-interface unbinding transition $A \,|||\, D \to A \,|\, B \,|\, C \,|\, D$ discussed above. This was treated in I using a "necklace" model in which "beads" of partially unbound interface segments of fluctuating length n_B are represented by the reunions of three vicious walkers. The beads alternate with "string" segments of tightly bound or merged interface representing the microscopic $A \,|||\, D$ interface with a "bare," reduced tension σ_{AD}. As explained, the nature of the transition is controlled by the exponent ψ_3. For $\psi_3 > 2$, as found above, the free energy increment on the bound side of the transition varies with $t \propto (T_{cW} - T) > 0$ as[1]

$$\Delta F \equiv F(T) - \Sigma_{\text{tot}}/k_B T = -\sum_{k=1} A_k t^k + A_s t^{\psi_3 - 1} + \cdots \tag{4.5}$$

with $A_1 > 0$. (For integral ψ_3 a factor $\ln t$ appears in the A_s term.[1])

Now, since $A_1 \neq 0$, this result describes a first-order transition with a latent heat (proportional to A_1): thus, feature (a) of the LN system has appeared. Furthermore, the singular correction which is controlled by ψ_3 in (4.5) suggests the presence of some type of criticality and an associated diverging length scale. As explained in I, the most direct definition of the longitudinal correlation length is via $\xi^{\parallel} = \overline{n_B^2}/\overline{n_B}$. This remains finite at $t = 0$, as expected for a first-order transition. However, if one considers the extended definition

$$\xi_{p,q}^{\parallel} = (\overline{n_B^p}/\overline{n_B^q})^{1/(p-q)} \tag{4.6}$$

one finds[4] a longitudinal correlation length *divergence* whenever $p > q > \psi_3 - 1$. The associated correlation exponent is readily found from I to be just

$$v_{p,q}^{\parallel} = 1 \tag{4.7}$$

[4] This and the other results quoted below follow, in the notation of I, from the formula

$$\overline{n_B^p} = \left[\left(z \frac{\partial}{\partial z} \right)^p G_B(z) \right] \Big/ G_B(z)$$

For $p < \psi_3 - 1$ this remains finite as $t \to 0+$; otherwise, it diverges with exponent $p - \psi_3 + 1$.

Evidently, there is only a single divergent length scale corresponding precisely to feature (b) of the LN subregime (C).

Lastly, one may enquire within the necklace model as to the transverse correlation lengths defined in (4.2). Since, as discussed in I, the transverse dimension of a single bead scales as $n_B^{1/2}$, which is just the law of diffusion, we expect $\xi_m^{\perp} \sim (\overline{n_B^{m/2}})^{1/m}$. On this basis one finds $\xi_m^{\perp} \sim t^{-\nu_m^{\perp}}$, where $\nu_m^{\perp}$ is given precisely by (4.4) provided one makes the identification

$$m_c(V_0) = 2\psi_3(b_j) - 2 \tag{4.8}$$

Finally, therefore, features (a)–(c) of the LN interface-plus-wall system have been matched!

How can the apparently close correspondence between the two models be understood? To this end, let us focus on the outermost walkers in a bead at separation $x_{13} \equiv l$ and average over the position of the middle walker, which we may suppose is unobservable. Then, along the lines argued in I or by other routes (see, e.g., ref. 2), one may conclude that the two outer vicious walkers continue to diffuse as before, but as though subject to an additional repulsive force, entropically generated by the invisible presence of the middle walker. This new effective force has a potential $V(l)$ decaying as W_0/l^2 (as $l \to \infty$). The amplitude W_0 depends continuously on the b_j; however, it does not seem easy to calculate its functional form explicitly.

The $1/l^2$ variation is just that considered by Lipowsky and Nieuwenhuizen.[7] Furthermore, when l becomes small in the necklace model the three walkers merge into a single string segment; thus, the tension difference $\sigma_{AD} - \sigma_{\text{tot}}$ corresponds to the short-range attractive amplitude U_0 in the LN system. As we saw in (4.3) and (4.4), the LN exponents depend continuously on $V_0 = W_0$ through $m_c(V_0)$. Via the correspondence (4.8), therefore, we should now expect ψ_3 to vary continuously with $W_0(b_j)$; hence, the dependence of ψ_3 on the b_j is no longer so surprising!

In conclusion, the continuous variation of ψ_3 may be understood as an intimate reflection of the marginality of the $1/l^2$ potential in interface unbinding in $d = 2$ dimensions. Since there must be a similar entropic repulsion varying asymptotically as $1/(x_p - x_1)^2$ for any $p \geqslant 3$ vicious walkers, it seems likely that ψ_p will depend continuously on the b_j $(j = 1,..., p)$ for all $p \geqslant 3$. Explicit calculations for four or more dissimilar vicious walkers would serve to check this speculation.

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